Griffiths

5.3. Show spherical Harmonics are either even or odd:

$$P: Y_l^m(\theta,\phi) \to (-1)^l Y_l^m(\theta,\phi).$$

Considered the associated Legendre func.

$$P_l^{m_l}(z) = \frac{1}{2^l l!}(1-z^2)^{m_l/2}\left(\frac{d}{dz}\right)^{(l+m_l)}(z^2-1)^l$$

This function is even in z except for $\left(\frac{d}{dz}\right)^{(l+m_l)}$:

$$P: \left(\frac{d}{dz}\right)^{l+m_l} \to (-1)^{l+m_l}\left(\frac{d}{dz}\right)^{l+m_l}.$$

$$\Rightarrow P: P_l^{m_l}(z) \to (-1)^{l+m_l} P_l^{m_l}.$$

By definition, $Y_l^{m_l} = \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m_l)!}{(l+m_l)!}}(-1)^{m_l} P_l^{m_l}(\cos\theta)e^{im_l\phi}$.

Under Parity, $Y_l^{m_l} \to (-1)^{l+m_l} Y_l^m = (-1)^l(-1)^{m_l} Y_l^m$.

* The $(-1)^{m_l}$ forms $(-1)^{2m_l} = 1$ with the $(-1)^{m_l}$ inside $Y_l^{m_l}$,

and now the sign of Y is only dependent on l.